DETERMINATION OF FREE MOTION PERIODS OF SKELETON-TYPE BUILDINGS WITH STRAIGHT LINEARIZATION METHOD

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Free nonlinear motions of a skeleton-type building have been studied, the dynamic design scheme of which appear in the form of systems with a finite numbered degree of freedom. The straight linearization method has been used. Values of nonlinear motion periods have been obtained for different real dependencies of the restoring force upon displacement.

Key words: Skeleton-type building, oscillation, linearization.

Most frequently applied methods for researching free (independent) non-linear motions are the following; perturbation method, asymmetric method of Krilov-Bogoliubov, Galergin method, etc. [3,6,7]. The systems with one degree of freedom are studied most thoroughly. The works dedicated to non-linear oscillations with many degrees of freedom are quite few. Mainly, the oscillations with two degrees of freedom are studied. Individual multimass systems using potential function and applying the properties of geodesic lines in space are studied in the works and particular results are obtained. Values of free motion periods of multi-storey buildings when the dependence of the restoring force on the floor displacement has the form of a cubic parabola are obtained in the works [4] using asymmetric method. Values of non-linear oscillation periods in case of the power dependence of the restoring force on the displacement are obtained in the work [5]. Some specific results are also shown in the works [3, 6, 8, 9] and they include, especially [9], detailed analysis of the existing methods for researching the non-linear systems. The work [7] includes the method of straight linearization and its essence is replacement of the specified non-linear system by the linear one. At the same time, the condition of the minimum of the quadratic deflection between the restoring forces of specified non-linear and obtained linear systems is used.

The method of straight linearization in this work covers the systems with many degrees of freedom. It is shown, that the computation simplicity of this method allows to get the values of free motion periods in an evident shape for many actually important cases of the dependences of the restoring force on the oscillation. The system oscillations with one degree of freedom for which new results on the non-linear oscillation periods are obtained were researched as well.

Let's review free motions of one-storey frame with rigid girder the design diagram of which is presented in form of the system with one degree of freedom (figure 1).

Equation of free motion systems when the dependence of the restoring force on the displacement is non-linear can be presented as follows:

$$m\frac{d^2y}{dt^2} + af(y) = 0\,, \tag{1}$$

where y – oscillation; m – lumped inertia; $R(y) = af(y)$ - restoring force; a – initial stiffness of the system; $a = tg\varphi$ - (figure 2); $f(y)$ - function characterizing the dependence of the restoring force on the displacement.

In a linear nonrigid system $f(y)$=y.

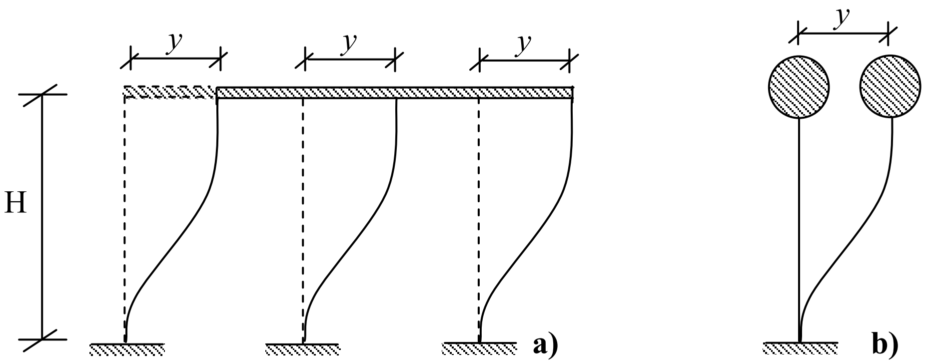
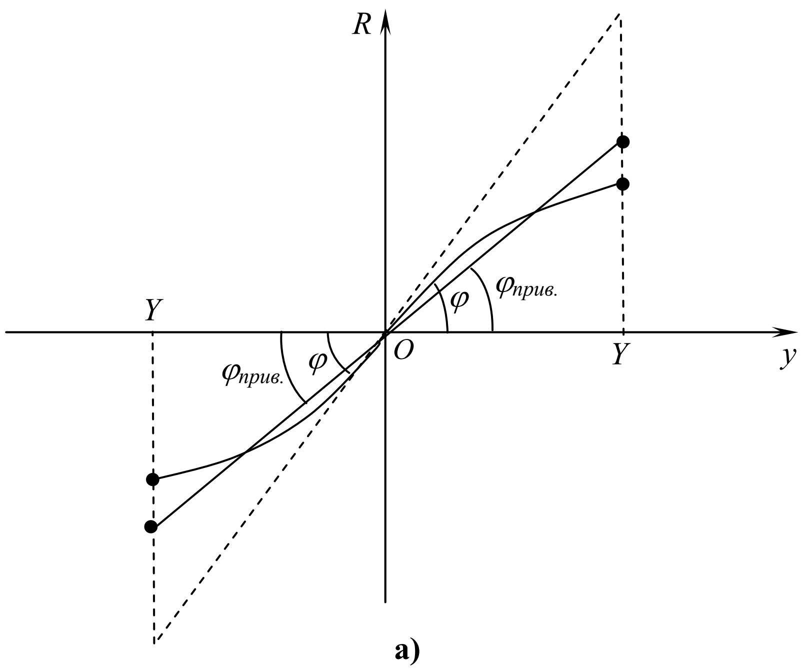
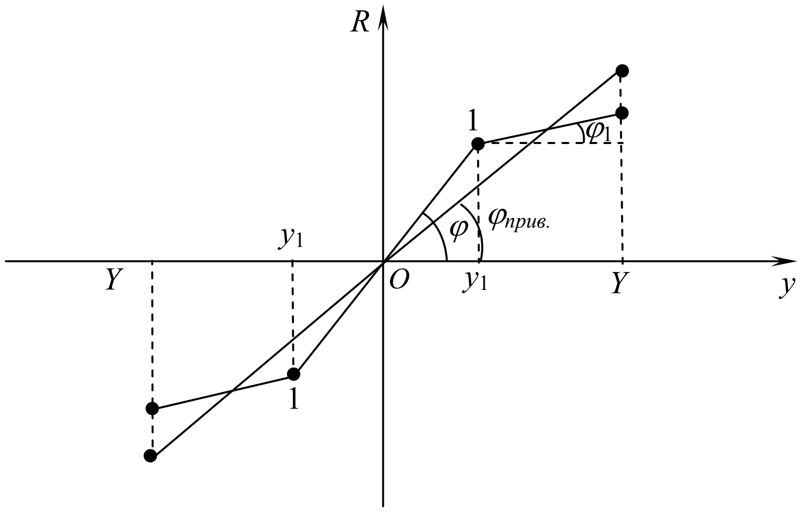


Figure 1. a – Frame Diagram; b – Dynamic Design Diagram



a)



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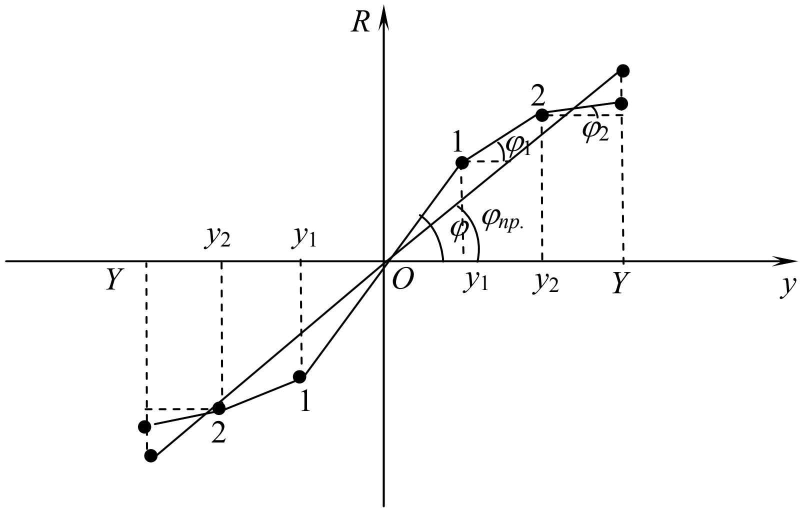

c)

Figure 2. Dependence of Restoring Force on Displacement. a – Curvilinear; b – Bilinear; c – Three-linear

Initial horizontal stiffness of the system which at the same time is the stiffness of the respective linear system is determined by formula [6]:

$$\mathbf{a} = \frac{\mathbf{12}}{\mathbf{H}^3}\sum_{j=1}^{N} E\mathbf{J}_j \ , \quad (2)$$

where E – module of column flexibility; J_j - moment of inertia of j column cross-section; N – number of frame columns (on figure 1 N = 3).

Let's assume, that specified (averaged) stiffness of $a_{пр}$ non-linear system is found, than the equation of free motions of the specified systems will have the following form:

$$m\frac{d^2y}{dt^2}+a_{пр}\cdot y=0. \qquad (3)$$

The variety of the restoring forces $r(y)$ of non-linear and linear system

$$r(y)=a\cdot f(y)-a_{пр}\cdot y. \qquad (4)$$

In order to emphasize the significance of deviations in large values y, $r(y)$ variety is taken in form of:

$$r(y)=[a\cdot f(y)-a_{пр}\cdot y]\cdot y. \qquad (5)$$

Problem is reduced to the integral minimization:

$$I=\int_{-Y}^{Y}[a\cdot f(y)-a_{пр}\cdot y]^2\cdot y^2dy. \qquad (6)$$

We have:

$$\frac{dI}{da_{пр}}=-2\int_{-Y}^{Y}[a\cdot f(y)-a_{пр}\cdot y]\cdot y^3dy=0. \qquad (7)$$

We obtain:

$$a_{пр}=\frac{\int_{-Y}^{Y}a\cdot f(y)y^3dy}{\int_{-Y}^{Y}y^4dy}. \qquad (8)$$

Taking into the account that there are even functions under the integral, we find:

$$a_{пр}=\frac{5a}{Y^5}\int_{0}^{Y}f(y)y^3dy. \qquad (9)$$

Frequency of the specified system which at the same time is the frequency of non-linear system is determined by the formula:

$$\omega_{прив}=\sqrt{\frac{a_{пр}}{m}}. \qquad (10)$$

By placing the value (9) into (10), we will get:

$$\omega_{прив}=\omega_{нел.}=\sqrt{\frac{a}{m}}\cdot\left\{\frac{5}{Y^5}\int_{0}^{Y}f(y)y^3dy\right\}^{1/2}=\omega_{лин.}\cdot\left\{\frac{5}{Y^5}\int_{0}^{Y}f(y)y^3dy\right\}^{1/2}. \qquad (11)$$

Non-linear system period is identified by formula:

$$T_{прив} = T_{нел.} = 2\pi\sqrt{\frac{m}{a_{пр}}} = 2\pi\sqrt{\frac{m}{a}}\cdot\left\{\frac{Y^5}{5a\int_0^Y f(y)y^3dy}\right\}^{1/2} = T_{лин.}\cdot\left\{\frac{Y^5}{5a\int_0^Y f(y)y^3dy}\right\}^{1/2}. \quad (12)$$

Let's discuss a case when the restoring force changes under the law of part of cubic parabola having the form of:

$$f(y) = y - \varepsilon y^3; \qquad \varepsilon \le \frac{1}{3y^2}. \quad (13)$$

In such case, from (9) and (10) will obtain:

$$\omega_{пр} = \omega_{лин}\left(1 - \frac{5}{7}\varepsilon Y^2\right)^{1/2}. \quad (14)$$

For case (13) there is an exact solution expressed through the elliptical integral of the first type (for instance, [6]).

In $\varepsilon Y^2 = 1/3$ $T_{нел.} = 1.146T_{лин.}$ and $1.154T_{лин}$. [6] the value of the exact and approximate periods varies by just 0.7%.

In bilinear diagram (figure 2, b) the function $f(y)$ will be written in form of:

$$f(y) = \begin{cases} y; & y \le y_1; \\ \gamma y + (1-\gamma)y_1; & y \ge y_1, \end{cases} \quad (15)$$

where $\gamma = a_1/a$.

From (11) will obtain:

$$\omega_{нел.} = \omega_{лин.}\left\{\gamma + \frac{5}{4}(1-\gamma)\frac{y_1}{Y} - \frac{(1-Y)}{4}\left(\frac{y_1}{Y}\right)^5\right\}^{1/2}. \quad (16)$$

In $\gamma = \frac{a_1}{a} = \frac{1}{4}$; $\beta = \frac{Y}{y_1} = 2$. From (16) will obtain:

$$\omega_{нел.} = 0.844\omega_{лин.}; \qquad T_{нел.} = T_{лин.}\cdot 1.184.$$

Exact value of the period in the above parameters [6]

$$T_{нел.} = T_{лин.}\cdot 1.193.$$

The difference makes 0.75%.

In case of the dependence of the restoring force on the displacement consisting of three rectilinear sections (figure 2, c) the function $f(y)$ is presented in form of:

$$f(y) = \begin{cases} y; & y \le y_1 \\ \gamma_1 y + (1-\gamma)y_1; & y_1 \le y \le y_2 \\ \gamma_2 y + (1-\gamma_1)y_1 + (\gamma_1 - \gamma_2)y_2; & y \ge y_2 \end{cases}, \quad (17)$$

where $\gamma_1 = \frac{a_1}{a}$; $\gamma_2 = \frac{a_2}{a}$; $a = tg\varphi$; $a_1 = tg\varphi_1$; $a_2 = tg\varphi_2$ (figure 2, c).

In such case, from (11) will have:

$$\omega_{нел.} = \omega_{лин.}\left\{\gamma_2 + \frac{5}{4}[1-\gamma_1+(\gamma_1-\gamma_2)\beta]\frac{y_1}{Y} - \left[\frac{1}{4}(1-\gamma_1)+\frac{1}{4}(\gamma_1-\gamma_2)\beta^5\right]\left(\frac{y_1}{Y}\right)^5\right\}^{1/2}, \quad (18)$$

Where Y – initial displacement $(Y > y_2)$; $\beta = \frac{y_2}{y_1}$.

It is not difficult to obtain the value of frequency of nonlinear oscillations when the dependence R-y is presented in form of the curve consisting of many rectilinear sections. It should be noted that it is either complicated or impossible to get this frequency in an evident form.

Let's review the case when the dependence R-y is presented in form of arbitrary power series. We have:

$$f(y) = \sum_{j=1}^{\infty} b_j y^j, \quad (19)$$

where b_j - defined coefficients of series.

The frequency of non-linear oscillations in this case is in form of:

$$\omega_{нел.} = \omega_{лин.}\sum_{j=1}^{\infty}\frac{5b_j}{j+1}y^{j-1}. \quad (20)$$

From (20) the frequency value can be obtained in case of (13) and also other versions described in [3,7].

Let's determine the frequency value when the R-y dependence is presented in form of ascending part of sinusoid.

The function $f(y)$ is given in form of:

$$f(y) = \frac{1}{\varepsilon}\sin\varepsilon y; \qquad \varepsilon \le \frac{\pi}{2y}. \quad (21)$$

The frequency of non-linear oscillation

$$\omega_{нел.} = \omega_{лин.}\left\{\left(\frac{15}{\varepsilon^3Y^3} - \frac{30}{\varepsilon^5Y^5}\right)\sin\varepsilon Y - \left(\frac{15}{\varepsilon^2Y^2} - \frac{30}{\varepsilon^4Y^4}\right)\cos\varepsilon Y\right\}^{1/2}. \quad (22)$$

Some functions such as cubic parabola or sinusoid are often used in researching the non-linear oscillations. However, these functions are limited by the forms (13), (21) that are associated with the fact that just an ascending part of these functions are used. This reduces the possibility of practical application of these dependences, as the value of ε coefficients included in these formulas should be small. From this point of view, it is suitable to use the functions having just the ascending part, i.e. their derivatives are always positive. One of such functions is arctangent function. Let's present the function $f(y)$ in form of:

$$f(y) = \frac{1}{\varepsilon}\mathrm{arctg}\,\varepsilon y. \quad (23)$$

In this case, for frequency will write:

$$\omega_{нел.} = \omega_{лин.}\left\{\frac{5}{4}\frac{\mathrm{arctg}\,\varepsilon Y}{\varepsilon Y} - \frac{5}{12\varepsilon^2Y^2} + \frac{5}{4\varepsilon^4Y^4} - \frac{5}{4\varepsilon^5Y^5}\mathrm{arctg}\,\varepsilon Y\right\}^{1/2}. \quad (24)$$

Using the formula (24) the values $\frac{\omega_{нел.}}{\omega_{лин.}}$ and then $\frac{T_{нел.}}{T_{лин.}}$ are deducted in various εY values. Calculation results are given in Table 1.

Values $\frac{\omega_{нел.}}{\omega_{лин.}}$ and $\frac{T_{нел.}}{T_{лин.}}$

Table 1

εY	$\frac{\omega_{нел.}}{\omega_{лин.}}$	$\frac{T_{нел.}}{T_{лин.}}$
1	0.913	1.095
2	0.789	1.267
4	0.625	1.600
5	0.570	1.754
6	0.531	1.883
8	0.469	2.134
10	0.424	2.360

As seen from table 1, the frequencies of the non-linear oscillations in increasing εY decrease in comparison with the frequencies of the linear oscillations, and the periods increase.

Let's now review free non-linear motions of multi-storey frames with rigid girders the design diagram of which is presented in form of the system with finite number of the degrees of freedom (figure 3).

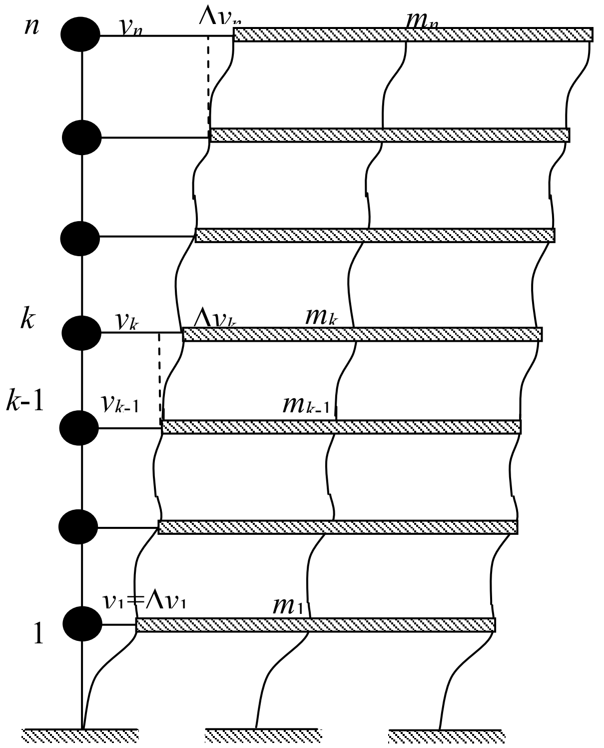

Figure 3. Diagram of Building Deformation

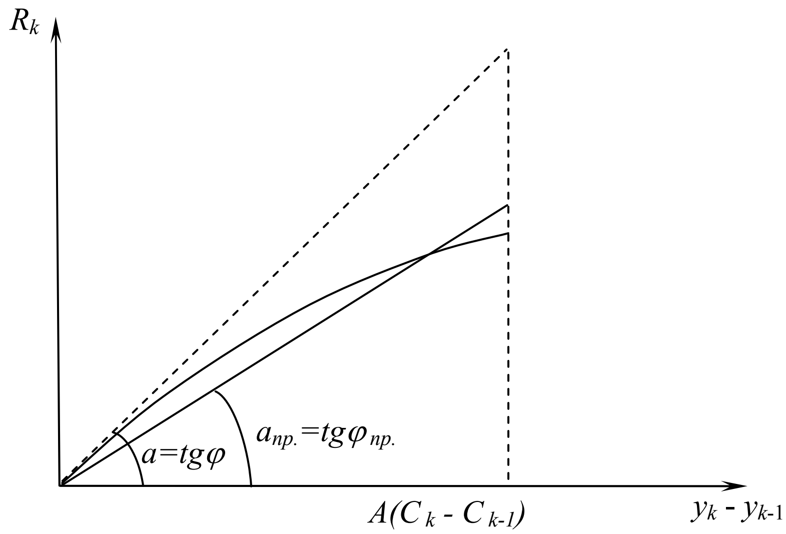

Figure 4. Dependence of Restoring Force on Floor Skewing

Equations of the free motions of non-linear nonrigid system can be written in form of [6]:

$$\sum_{i=k}^{n} m_i \frac{d^2 y_i}{dt^2} + R_k(y_k - y_{k-1}) = 0\,; \quad k = 1,2,...,n\,, \tag{25}$$

where y_k - displacement of *k* floor, m_i - inertia lumped at the level of *k* floor; $R_k(y_k - y_{k-1})$ - restoring force of $R_k(y_k - y_{k-1})$ *k* floor; n – number of floors.

In future we will accept that the hardness of all the floors is equal. The restoring force of *k* floor will be:

$$R_k(y_k - y_{k-1}) = af_k(y_k - y_{k-1})\,, \quad k = 1,2,...,n\,, \tag{26}$$

where a – initial floor hardness.

We accept that the non-linear system can be replaced by the linear one with the hardness $a_{пр.}$ In such case the restoring force of the floor will be:

$$R_k(y_k - y_{k-1}) = a_{пр.}(y_k - y_{k-1})\,. \tag{27}$$

Let's put together the differential of the restoring forces of the specified non-linear and linear systems. We accept that the coefficients of the oscillation form of the nonlinear system are proportional to the coefficients of the forms of the linear system. Like in (6), in this case the problem is reduced to the minimization of the integral representing mean-square residual of the restoring forces of the non-linear and linear systems:

$$I = \sum_{k=1}^{n} 2 \int_0^{Y(C_k - C_{k-1})} [af_k(y_k - y_{k-1}) - a_{пр.}(y_k - y_{k-1})]^2 (y_k - y_{k-1})^2 d(y_k - y_{k-1})\,, \tag{28}$$

where C_k - specified coefficients of basic oscillation forms.

Let's identify $\Delta_k = (y_k - y_{k-1})$; $\overline{\Delta}_k = C_k - C_{k-1}$.

From (28) we have:

$$\frac{dI}{da_{пр.}} = -4\sum_{k=1}^{n}\int_{0}^{\overline{\Delta}_k Y} af_k(\Delta_k)\cdot\Delta_k^3 d\Delta_k + 4\sum_{k=1}^{n}\int_{0}^{\overline{\Delta}_k Y}\Delta_k^4 d\Delta_k = 0\cdot \quad (29)$$

From here will obtain:

$$a_{пр.} = \frac{5a}{Y^5}\frac{\sum_{k=1}^{n}\int_{0}^{\overline{\Delta}_k Y} f_k(\Delta_k)\cdot\Delta_k^3 d\Delta_k}{\sum_{k=1}^{n}\overline{\Delta}_k^5}\,. \quad (30)$$

The oscillation frequency of the major tone of the system with $a_{пр.}$ the hardness and m lumped inertia will be written in such a way:

$$\omega_{пр.}^2 = \omega_{нел.}^2 = \frac{a_{пр.}}{m}\lambda_1\,, \quad (31)$$

where λ_1 - coefficient depending on number of floors. Placing (30) in (31) will obtain:

$$\omega_{пр.}^2 = \frac{a}{m}\lambda_1\cdot\frac{5a}{Y^5}\frac{\sum_{k=1}^{n}\int_{0}^{\overline{\Delta}_k Y} f_k(\Delta_k)\cdot\Delta_k^3 d\Delta_k}{\sum_{k=1}^{n}\overline{\Delta}_k^5}\,. \quad (32)$$

Since $\frac{a}{m}\lambda_1 = \omega_{лин.}^2$ from (32) will obtain:

$$\omega_{нел.} = \omega_{лин.}\left\{\frac{5a}{Y^5}\frac{\sum_{k=1}^{n}\int_{0}^{\overline{\Delta}_k Y} f_k(\Delta_k)\cdot\Delta_k^3 d\Delta_k}{\sum_{k=1}^{n}\overline{\Delta}_k^5}\right\}^{1/2}. \quad (33)$$

$\omega_{нел.}$ values in case of the same dependencies that were researched for the system of one degree of freedom can be identified by the formula (33).
Let's review various cases of the non-linear laws of the restoring force variation.

$$1.\ f_k(\Delta_k) = \Delta_k - \varepsilon\Delta_k^3\,. \quad (34)$$

From (33) will obtain:

$$\omega_{нел.} = \omega_{лин.}\left\{\frac{\sum_{k=1}^{n}\left[\overline{\Delta}_k^5 - \frac{5}{7}\varepsilon Y^2\right]\sum_{k}^{n}\overline{\Delta}_k^7}{\sum_{k=1}^{n}\overline{\Delta}_k^5}\right\}^{1/2}. \quad (35)$$

$$2.\ f_k(\Delta_k) = \sum_{j=1}^{\infty} b_j\Delta_k^j\,. \quad (36)$$

For $\omega_{нел.}$ will get:

$$\omega_{нел.} = \omega_{лин.} \left\{ \frac{5\sum_{k=1}^{n}\sum_{j=1}^{\infty} b_j \cdot \frac{Y^{j-1}\overline{\Delta}_k^{j+4}}{j+4}}{\sum_{k=1}^{n}\overline{\Delta}_k^5} \right\}^{1/2} . \quad (37)$$

$$3.\ f_k(\Delta_k) = \frac{1}{\varepsilon}\sin\varepsilon\Delta_k . \quad (38)$$

$$\omega_{нел.} = \omega_{лин.} \left\{ \frac{5\sum_{k=1}^{n}\left[\frac{3\overline{\Delta}_k}{\varepsilon^3 Y^3} - \frac{6}{\varepsilon^5 Y^5}\right]\sin\varepsilon Y\overline{\Delta}_k - \left[\frac{\overline{\Delta}_k^3}{\varepsilon^2 Y^2} - \frac{6\overline{\Delta}_k}{\varepsilon^4 Y^4}\right]\cos\varepsilon Y\overline{\Delta}_k}{\sum_{k=1}^{n}\overline{\Delta}_k^5} \right\}^{1/2} . \quad (39)$$

$$4.\ f_k(\Delta_k) = \begin{cases} \Delta_k; & \Delta_k = \Delta_{k1}; \\ \gamma\Delta_k + (1-\gamma)\Delta_{k1}; & \Delta_k \geq \Delta_{k1}. \end{cases} \quad (40)$$

$$\omega_{нел.} = \omega_{лин.} \left\{ \frac{\sum_{k=1}^{n}\left[-\frac{(1-\gamma)}{4}\frac{\Delta_{k1}^5}{Y^5} + \frac{\gamma - \Delta_k^5}{4}\frac{(1-\gamma)\Delta_{k1}\Delta_k^4}{Y}\right]}{\sum_{k=1}^{n}\overline{\Delta}_k^5} \right\}^{1/2} . \quad (41)$$

$$5.\ f_k(\Delta_k) = \begin{cases} \Delta_k; & \Delta_k = \Delta_{k1}; \\ \gamma_1\Delta_k + (1-\gamma_1)\Delta_{k1}; & \Delta_{k1} \leq \Delta_k \leq \Delta_{k2}; \\ [\gamma_2\Delta_k + (1-\gamma_1)\Delta_{k1} + (\gamma_1 - \gamma_2)\Delta_{k2}]; & \Delta_k \geq \Delta_{k2}; \end{cases} \quad (42)$$

$$\omega_{нел.} = \omega_{лин.} \left\{ \frac{\gamma_2 + \frac{5}{4}[1-\gamma_1 + (\gamma_1-\gamma_2)\beta]\frac{\Delta_{k1}}{\overline{\Delta}_k Y} - \left[\frac{1}{4}(1-\gamma_1) + \frac{1}{4}(\gamma_1-\gamma_2)\beta^5\right]\left(\frac{\Delta_{k1}}{\overline{\Delta}_k Y}\right)^5}{\sum_{k=1}^{n}\overline{\Delta}_k^5} \right\}^{1/2} . \quad (43)$$

$$6.\ f_k(\Delta_k) = \frac{1}{\varepsilon}\operatorname{arctg}\varepsilon\Delta_k . \quad (44)$$

$$\omega_{нел.} = \omega_{лин.} \left\{ \frac{\sum_{i=1}^{n}\frac{5}{4\varepsilon Y}\overline{\Delta}_k^4\operatorname{arctg}\varepsilon Y\overline{\Delta}_k - \frac{5}{12\varepsilon^2 Y^2}\overline{\Delta}_k^3 + \frac{5}{4\varepsilon^4 Y^4} - \frac{5}{4\varepsilon^5 Y^5}\operatorname{arctg}\varepsilon Y\overline{\Delta}_k}{\sum_{k=1}^{n}\overline{\Delta}_k^5} \right\}^{1/2} . \quad (45)$$

In case of the formula (45), the values $\omega_{нел.}/\omega_{лин.}$ and then $\omega_{нел.}/\omega_{лин.}$ for n = 5 are calculated. The obtained values C_k, $\overline{\Delta}_k$ $(k = 1,2,...5)$ taken from [6] are provided in Table 2.

Table 2

k	C_k	$\overline{\Delta}_k$
1	**0.2856**	**0.2856**
2	**0.548**	**0.2624**
3	**0.7656**	**0.2176**
4	**0.922**	**0.1564**
5	**1**	**0.078**

The values $T_{nonlinear}/T_{linear}$ in case of $\varepsilon Y = 2,4,6,8,10$ were respectively 1.04; 1.1; 1.194; 1.285; 1.381. It is interesting to compare $T_{nonlinear}/T_{linear}$ obtained in case of $n = 1$ and $n = 5$. Respective dependences are given in Figure 5.

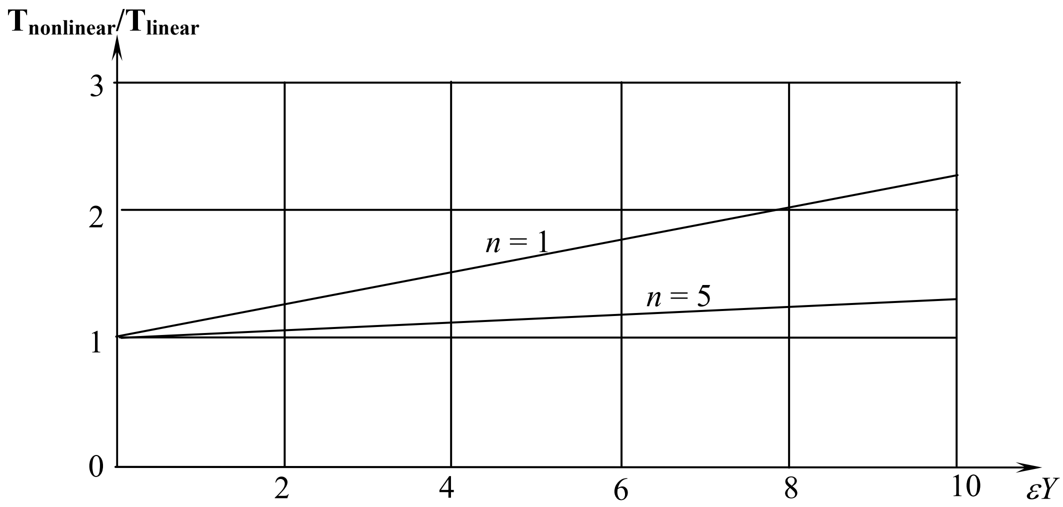

Figure 5. Dependences of Period Relation on Displacement

As seen from this diagram, $T_{nonlinear}/T_{linear}$ increases together with increasing εY. At the same time, in case of the same displacement εY, the relation $T_{nonlinear}/T_{linear}$ is more in one-storey building.

In all the reviewed cases of non-linear deformation the loading and unloading occurred under the same linear law. If the loading occurs under the non-linear law and the unloading under the linear law, than the period of hysteretic oscillations $T_{гис.}$ can be determined by formula [4].

$$T_{гис.} = \frac{T_{нел.} + T_{лин.}}{2}, \qquad (46)$$

where $T_{nonlinear}/T_{linear}$ – respectively the period of the non-linear and linear oscillations.

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